Derivatives!

- what is a derivative?
  - The instantaneous rate of change of a function with respect to a variable.
    (kind of like the slope at a single point)

- How do we find the rate of change of a function?

Let $f(x)$ be a function.

The slope from X_1 to X_2 is given by

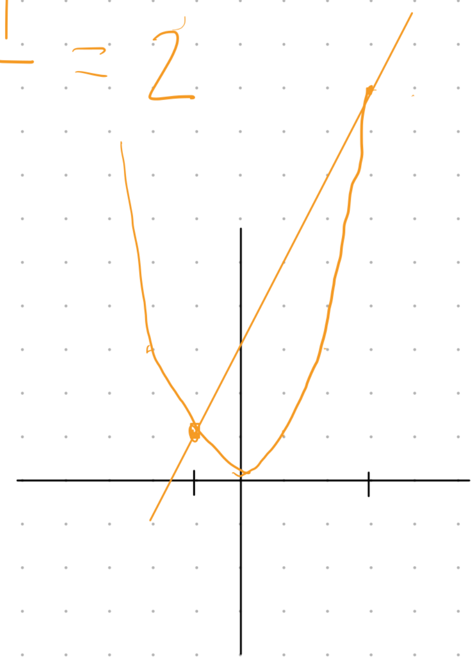
$$\frac{\Delta f(x)}{\Delta X} = \frac{f(X_2) - f(X_1)}{X_2 - X_1}$$

Imagine $f(x) = x^2$.

Say we want to find the rate of change at $x = -1$.

To calculate the slope we need another point. I'll use $x = 3$.

$$\frac{3^2 - (-1)^2}{3 - (-1)} = \frac{9 - 1}{3 + 1} = 2$$



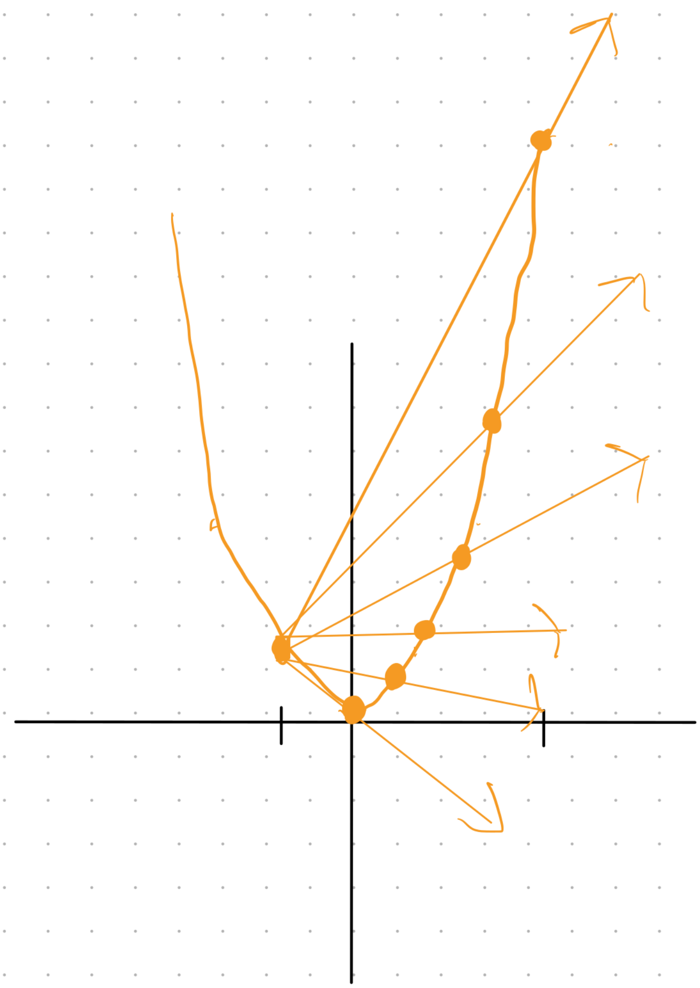
Clearly, 2 is a bad estimation for the rate of change at $x = -1$. The function is decreasing at $x = -1$.

We need to select another value for x_2 to calculate an accurate rate of change at $x = -1$.

We can generalize the slope equation by setting $x_2 = x_1 + h$. (for some $h > 0$)

now: $$\frac{f(x_1+h) - f(x_1)}{(x_1+h) - x_1} = \frac{f(x_1+h) - f(x_1)}{h}$$

Note: generally, the smaller h gets, the closer the slope gets to the instantaneous rate of change at x_1.



To find the instantaneous rate of change we want h to be as small as possible. Note: we can't set $h=0$ since we can't divide by 0.

We use the <u>limit</u> to get as close to 0 as possible.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In our example with $f(x) = x^2$:

$$\lim_{h \to 0} \frac{(x+h)^2 - (x)^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh}{h} + \frac{h^2}{h}$$

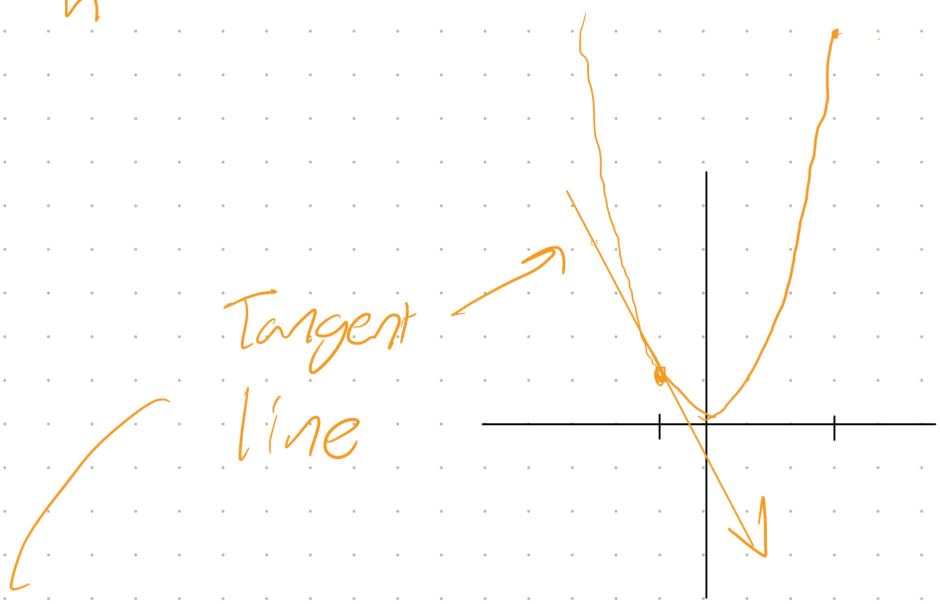
$$= \lim_{h \to 0} 2x + h$$

$$= 2x + 0$$

$$= 2x$$

$@ \; x = -1$

$$\lim_{h \to 0} \frac{(x+h)^2 - (x)^2}{h} = 2(-1) = -2$$



The slope of the tangent line is equal to the instantaneous rate of change at any given point.

Def: The derivative of $f(x)$ with respect to x is defined by:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

A derivative is denoted by:

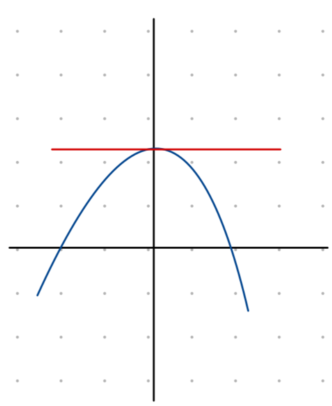
$$f'(x) \quad \text{or} \quad \frac{df}{dx} \quad \text{or} \quad \frac{df(x)}{dx} \quad \text{or} \quad \frac{d}{dx}f(x)$$

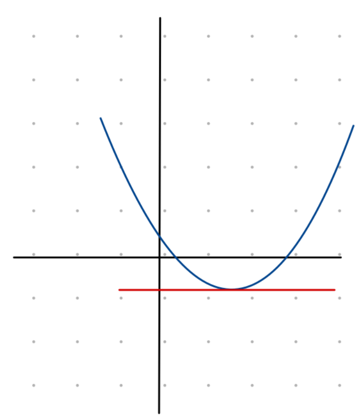
The derivative of a function gives the slope of the tangent line at a given point.

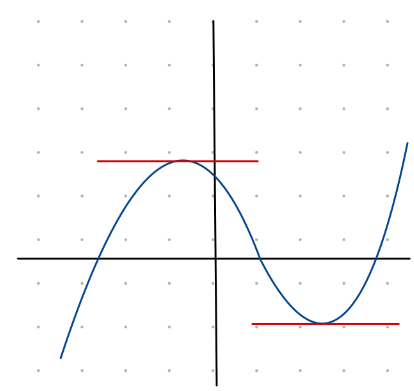
Why is this helpful?

In microeconomics we are basically always solving optimization problems. (either <u>min</u> or <u>max</u>)

Lets look at some functions:





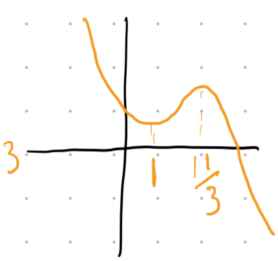


note: the tangent lines at all local minimums and maximums have a flat tangent line (slope = 0).

We can use this information to solve optimization problems by setting the derivative = 0.

Another note: you can check if the value you find is at a local min or max by taking the 2nd derivative.

2nd derivative positive: local min



2nd derivative negative: local max

ex: $y = x + x^2 - (x-2)^3$

$y' = 1 + 2x - 3(x-2)^2 = 1 + 2x - 3x^2 + 12x - 12 = -3x^2 + 14x - 11$

$0 = -3x^2 + 14x - 11 \rightarrow \boxed{x = 1, \frac{11}{3}}$

$y'' = -6x + 14$ $\quad y''(1) = 8$ $\quad y''(\frac{11}{3}) = -8$

local min $\quad$ local max

Taking the limit of a function takes a lot of effort. There often is a faster way using derivative rules.

your best friend

$$\boxed{\begin{array}{l}\textbf{Power Rule}\\ \frac{d}{dx}(x^n) = n x^{n-1}\end{array}}$$

ex:

$$f(x) = x^4 + 3x^3 - 2x^2 + 5x - 2$$

$$f'(x) = 4x^3 + 3 \cdot 3x^2 - 2 \cdot 2x^1 + 5 - 0$$

$$f'(x) = 4x^3 + 9x^2 - 4x + 5$$

$$\frac{d}{dx}(c \cdot x^n) = c \cdot n x^{n-1}$$

$$\frac{d}{dx}(c \cdot x) = c$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(c) = 0$$

note: anything that isn't x (or the variable you differentiate by) is treated as a constant.

Product Rule

$$\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

ex:

$$y = (x^2+7)(-x^3+2x-6)$$

$$\frac{dy}{dx} = (x^2+7) \cdot (-3x^2+2) + (2x) \cdot (-x^3+2x-6)$$

$$= -3x^4 + 2x^2 - 21x^2 + 14 - 2x^4 + 4x^2 - 12x$$

$$\frac{dy}{dx} = -5x^4 - 15x^2 - 12x + 14$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

"low dhigh minus high dlow over the square of whats below"

ex: $y = \dfrac{x^2+2x}{4x-3}$

$$\frac{dy}{dx} = \frac{(4x-3)(2x+2) - (x^2+2x) \cdot 4}{(4x-3)^2}$$

$$\frac{dy}{dx} = \frac{8x^2+8x-6x-6-4x^2-8x}{(4x-3)^2}$$

$$\frac{dy}{dx} = \frac{4x^2-6x-6}{(4x-3)^2}$$

Chain Rule

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

ex:

$(6x+4)^5$

$\rightarrow 5(6x+4)^4 \cdot 6$

Other derivatives to note:

$$\frac{d}{dx}(\log x) = \frac{1}{x} = x^{-1}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = a^x \log(a)$$